

Tilburg University

A Theory of the Gambling Effect

Diecidue, E.; Schmidt, U.; Wakker, P.P.

Publication date:
2000

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Diecidue, E., Schmidt, U., & Wakker, P. P. (2000). *A Theory of the Gambling Effect*. (CentER Discussion Paper; Vol. 2000-75). Microeconomics.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Center
for
Economic Research

No. 2000-75

A THEORY OF THE GAMBLING EFFECT

By Enrico Diecidue, Ulrich Schmidt and Peter P. Wakker

August 2000

ISSN 0924-7815

A Theory of the Gambling Effect

Enrico Diecidue^t, Ulrich Schmidt^k, & Peter P. Wakker^{t *}

^t: *CentER, Tilburg University, The Netherlands*

^k: *Institut für Finanzwissenschaft und Sozialpolitik,
Christian-Albrechts-Universität zu Kiel, Germany*

September 28, 1999

Abstract

This paper presents a model for the “gambling effect,” i.e., the effect that risky gambles are evaluated differently than riskless outcomes due to an intrinsic utility (or disutility) of gambling. The model turns out to violate stochastic dominance and therefore its primary applications will be descriptive. It sheds new light on empirical observations of risk attitudes and provides new insights into the distinction between risky and riskless utility.

Running Title: Gambling effect

*corresponding author: Peter Wakker, CentER, Tilburg University, P.O. Box 90153, Tilburg, 5000 LE, The Netherlands, - - 31-13.466.30.45 (S), - - 31-13.466.32.80 (F), - - 31-71.532.39.81 (H), Wakker@MDM.MedFac.LeidenUniv.NL

Journal of Economic Literature Classification number: D81, C60

Keywords: certainty effect, gambling effect, risk aversion, nonexpected utility

1 Introduction

Many authors have suggested a “gambling effect,” i.e., the effect that people process risky choice options in a different manner than riskless ones. The effect may be due to an intrinsic utility for the presence or absence of risk, and underlies much of the commonly observed risk aversion (Royden, Suppes, & Walsh 1959). It may have confounded many empirical investigations into risk attitudes (Conlisk 1993). The effect was already mentioned by von Neumann & Morgenstern (1944, pp. 28, 629, 632). For an extensive survey of the literature, with extensive empirical evidence, see Conlisk (1993). Several historical citations are given by Pope (1995). Our paper presents a decision theory for the gambling effect, generalizing and simplifying earlier contributions by Fishburn (1980) and Schmidt (1998).

The gambling effect is intuitively appealing. It is psychologically plausible that people, in the presence of risk, invoke different evaluation methods than when no risk is present. Surprisingly, only two papers have provided preference theories for the gambling effect (Fishburn 1980, Schmidt 1998), to the best of our knowledge.¹ An explanation may be that any decision model of the effect necessarily violates stochastic dominance (see Observation 7). Such violations are normatively undesirable, hence the gambling effect is not interesting from a normative perspective. For many years, systematic violations of stochastic dominance were also held to be descriptively undesirable because they seem to be implausible. In addition, descriptive models should satisfy some minimal rationality conditions in order to be tractable and permit theoretical derivations. Stochastic dominance was usually considered to be one of those rationality conditions.

¹For decision under uncertainty, Luce & Marley 1999 give an alternative approach, based on a different paradigm for decision making.

The demonstrated violation of stochastic dominance allows for a new speculation on a text in von Neumann & Morgenstern (1944): ‘... concepts like a “specific utility of gambling” cannot be formulated free of contradiction on this level. ... But anybody who has seriously tried to axiomatize that elusive concept, will probably concur with it’ (p. 28 including footnote 3). The present paper is the third which has seriously tried to axiomatize the specific utility of gambling. It does not seem impossible to us that von Neumann & Morgenstern, when alluding to a “contradiction,” foresaw a violation of something as basic as stochastic dominance.² Remarkably, also Tversky (1967) is negative on the possibility to axiomatize the gambling effect (“In spite of its apparent appeal, this approach does not yield testable predictions.” p. 198).

In the last decades, more interest has arisen in basic violations of rationality. The preference reversal effect (Lichtenstein & Slovic 1971, Lindman 1971) was a first signal of rationality violations more basic than expected before. Framing effects (Tversky & Kahneman 1981) provided another signal. Finally, Tversky & Kahneman (1986) developed a clever example where violations of stochastic dominance can be generated systematically. Starmer & Sugden (1993) formalized the underlying “event splitting effect,” further refined and confirmed in several papers by Birnbaum and co-authors (e.g., Birnbaum & Navarrete 1998). These findings, the psychological plausibility of the gambling effect, and the interest in the phenomenon expressed by many, have led to the present paper. Let us now turn to the impact of the gambling effect on risk attitudes.

Under expected utility, a special preference for riskless outcomes is defined as risk aversion and modeled through a concave utility. Several generalizations have

²At other places, e.g., p. 632, von Neumann & Morgenstern suggest that reduction of compound lotteries should be abandoned so as to accommodate for a utility of gambling. Our paper does not consider compound gambles.

been proposed during the last decades, such as the certainty effect of prospect theory (Kahneman & Tversky 1979). Here probabilities are transformed nonlinearly into decision weights. The transformation function is assumed to decrease steeply at certainty 1, leading to additional preference for riskless outcomes such as exhibited in the Allais paradox and outside the realm of expected utility. Modern variations, through rank-dependent expected utility (Quiggin 1981), cumulative prospect theory (Luce 1991, Tversky & Kahneman 1992), and other models (Dekel 1986, Sugden 1993) have been proposed. All these models have in common that the special preference for riskless options is smooth, i.e., there is no qualitative difference between risky and riskless options but a gradual one. If lotteries are risky but close to riskless then their evaluation is also close to the riskless evaluation.

The gambling effect as modeled in this paper entails a more drastic form of the certainty effect, where the transition of certainty to risk is abrupt and discontinuous. Such a transition is psychologically plausible. As soon as a sure outcome is changed into a risky gamble, no matter how small the risk is, people may leave their “riskless” evaluation process and turn to their “risky” evaluation process instead. The gambling effect describes, in a way, a minimal deviation from expected utility to accommodate the Allais paradox. As long as no riskless options are involved, expected utility can be completely well satisfied. Only when riskless options are involved, a deviation from expected utility is required.

Many empirical investigations have suggested that violations of expected utility are primarily due to “boundary” effects, i.e., drastic changes in the evaluation process when the number of positive-probability outcomes is changed (Conlisk 1989, Sopher & Gigliotti 1993, Harless & Camerer 1994). Formal models describing such changes in evaluation have been proposed by Viscusi (1989), Luce (1999), Neilson (1992), and Humphrey (1998). In fact, already the often-discussed violation of stochastic

dominance by original prospect theory (Kahneman & Tversky 1979) was due to a similar effect. The gambling effect model formalizes the simplest and most prominent form of a boundary effect.

Different evaluations for risky and riskless options have been proposed in other contexts. For instance, Dyer & Sarin (1982), and many other papers (see the references in Wakker 1994, Section 2), have proposed that expected utility with a utility function u be used for risky choices, but a different function v , a value function, be used for riskless evaluations. Examples of riskless evaluations are intertemporal, welfare, or strength of preference judgments. This distinction between utility and value has been widely accepted (Dyer & Sarin 1982), although some authors have suggested modifications (Wakker 1994, Stalmeier & Bezembinder 1998). Our model takes the difference between risky and riskless evaluations one step further than the Dyer & Sarin model. For a preference between a riskless outcome x and a risky lottery P , the Dyer & Sarin model assumes that both x and P be evaluated through u and its expectation. Our model proposes that already in the choice between x and P , the value of x be $v(x)$ and not $u(x)$. Hence the Dyer & Sarin model is within the realm of expected utility but our model deviates from expected utility. Another deviation is that, contrary to the Dyer & Sarin model, we need not require that u and v order outcomes in the same manner.

2 Basic Ways to Model the Gambling Effect

In general, the gambling effect means that a preference $x \succsim P$ between a sure outcome x and a risky lottery P holds if and only if the following formula holds:

$$W(x) \geq W(P) - C(x, P). \quad (1)$$

W is a preference functional from some risk theory —it will be expected utility in this paper. C does not refer to an intrinsic value of x or P but instead describes a holistic cost for the presence or absence of gambling. Costs of gambling may contribute to risk aversion and insurance. Negative costs, so a positive utility of gambling, may underly public lotteries, horse race betting, and casinos.

Without further restrictions, the model does not have any implication for preference and even permits intransitivities. For instance, we can set $C(x, P) + W(x) - W(P)$ equal to 1 whenever x is preferred to P , equal to 0 whenever x is indifferent to P , and equal to -1 whenever P is preferred to x , thus accommodating any arbitrary preference relation. Hence, to generate empirically meaningful predictions, restrictions must be imposed on C .

Two approaches have been considered to ensure transitivity and extend the model to other choices than between a risky and a riskless gamble. In the first approach, C depends only on P , in the second C depends only on x . In the first approach the costs can be written as $C(P)$. Then transitivity is satisfied, with $W(P) - C(P)$ the value of the gamble and C the holistic cost of gambling. In the second approach, the costs can be written as $C(x)$. Transitivity is satisfied with $W(x) + C(x)$ the value of the riskless outcome and C the benefit of certainty. At the end of Section 6, we will demonstrate that the second approach can always be rewritten as a special case of the first approach, with $C(P)$ linear in P .

As common in the literature on the gambling effect, we assume that preferences between risky lotteries agree with expected utility. Hence the only deviation from expected utility considered is due to the gambling effect. Generalizations permitting also other deviations are left to future studies. A first suggestion was made by Tversky (1967). He considered a domain of gambles with one nonzero outcome and permitted nonlinear probability transformation there.

In the first approach, the model is too general if C can depend on the gamble P in any possible manner. Then, irrespective of W , the model can accommodate almost any transitive relation by taking C accordingly. Therefore Fishburn (1980), the only decision-theoretic work on the first approach that we are aware of, added further restrictions. These, however, turn out to imply that his model becomes a special case of the second approach. A detailed discussion is given in Section 3. Observation 7, demonstrating violation of stochastic dominance for the second approach, therefore also applies to Fishburn’s model.

Our paper is based on the second approach. Risky gambles P will be evaluated by expected utility with respect to a “von Neumann-Morgenstern” utility function u . Riskless outcomes x are evaluated by a “riskless” function $v(x)$ (equal to $W(x) + C(x)$ in the notation of Eq. 1). The model could have a normative interpretation as a model of transaction costs. For instance, if a sure outcome x could be collected right away but for any risky gamble a contract would have to be signed to settle conditional agreements, then C could designate the cost of the contract. An alternative example can be as follows. The outcome of any risky gamble could be collected only after the resolution of some uncertainty and be subjected to tax depending on that outcome, but a sure outcome x could be collected beforehand evading the tax. It can then be convenient to incorporate the tax saved under x not in the description of the outcome x , but in the function C .

3 Related Literature

This section discusses related works on the gambling effect in some detail. We first discuss the decision-theoretic models by Fishburn (1980), Schmidt (1998), and Luce & Marley (1999). The first approach described in the preceding section, i.e., Eq.

(1) with C depending only on P , was the starting point in Fishburn (1980). To obtain predictive power, Fishburn added further restrictions. We first discuss his central representation, Theorem 3. Fishburn assumed that not only $W - C$, but also W in isolation, represents preferences over risky gambles. This implies that C is a transform of W , i.e., the cost of gambling depends only on the preference value of the lottery and not on other characteristics. Further, the risky and riskless functionals order outcomes in the same manner, and W is assumed to be an expected utility functional. These assumptions also underlie the “fragmented” (Fishburn 1980, p. 441) representations in Fishburn’s Theorems 1 and 2 and in his other theorems. Under these assumptions, Fishburn’s model can be rewritten as a special case of the second approach with C depending on the outcome x . It is a special case because both W and $W - C$ represent preferences over risky lotteries, implying that v and u order outcomes in the same manner.³

The second approach, with C depending only on x , has been suggested by Tversky (1967) and Fishburn (1980) and was formalized by Schmidt (1998), a work written independently of Fishburn (1980). Schmidt used the term “certainty effect” instead of our general term gambling effect. We generalize Schmidt’s (1998) approach by permitting general outcomes as did Fishburn. Schmidt assumed a separable metric outcome space and imposed continuity and boundedness conditions. Thus, his work does not provide a genuine generalization of the von Neumann Morgenstern expected utility model.

Luce & Marley (1999) consider decision under uncertainty instead of risk. Un-

³In the notation of Fishburn’s Theorem 3, both u and $u + \phi$ order preferences over risky lotteries. Hence we can define a strictly increasing transformation f such that $f(u + \phi) = u$. Applying f to Fishburn’s representation $u + \phi$ yields the expected utility representation for lotteries, and for outcomes x , f transforms $(u + \phi)$ into what we call v .

certainty is described through events for which no probabilities need be given. Their model can be considered a special case of the first approach to Eq. (1) with the cost $C(x, P)$ depending on P only through the uncertain events used to describe P and not through the outcomes (with their “kernel equivalent” playing the role of our W). Their approach deviates from the common decision paradigm in several respects, e.g. through “joint receipts” (receiving more than one act simultaneously) and acts not being identified with mappings from states to outcomes.

We next discuss some works that formulated version of the gambling effect but did not provide preference axioms. Tversky (1967) explicitly pointed out that discrepancies between risky and riskless utilities can be affected by the gambling effect. He considered single nonzero outcomes and the logarithm of von Neumann-Morgenstern utility. Nonlinear probability weighting was permitted. Tversky did not elaborate on his interpretations and experimental measurements formally, and did not explicate their similarities and differences with Savage’s (1954) expected utility versus Edwards’ (1962) risk model. Hence his model is not discussed further.

Conlisk (1993) considered the first approach (C depends on P), for two-outcome gambles with expectation zero. He assumed that C is negative (so gambling is valued positively) but remains small, so that it can affect gambles for small outcomes but not for large outcomes. This model can explain risk seeking for small-outcome gambles. Empirical evidence was presented. Conlisk derived plausible implications from assumptions such as concavity on C and the other functions. This also suggests that gambles with real incentives, as commonly adopted in experimental economics, may be confounded by the gambling effect. Note that the Allais paradox cannot be explained by Conlisk’s model because C is negative (Conlisk 1993 end of Section 5). Section 5 also demonstrates that decreasing proportional risk aversion, often observed empirically, may be explained by a gambling effect.

Neilson (1992) and Humphrey (1998) considered models where for each natural number n , a utility function u_n is given, and gambles with exactly n positive-probability outcomes are evaluated by expected utility with respect to u_n . They gave empirical evidence supporting such models. The gambling effect model studied in the present paper can be considered the special case of the Neilson-Humphrey model where only u_1 deviates from the other utilities and all u_j s for $j \geq 2$ are identical. This special case does cover the main cause for boundary effects, i.e., the certainty effect. Our paper can be interpreted as a theoretical foundation for the simplest case of the Neilson-Humphrey model.

Another general model is presented by Le Menestrel (1999), where the evaluation of a gamble can depend on the process generating the gamble (see also Grant, Kajii, & Polak 1998). Le Menestrel shows that the gambling effect is a special case of his general model and demonstrates that this special case can have empirical implications.

For health outcomes, Richardson (1990) considered two-outcome gambles and assumed that C consists of a constant utility of gambling plus a term depending on the sure outcome x . Other references from health economics are Bombardier et al. (1982), Gafni & Torrance (1984), Loomes (1993), Stiggelbout et al. (1994). In the health domain, Gafni and colleagues argued for a systematic difference between risky and riskless options (Mehrez & Gafni 1989, Gafni & Birch 1997). Their theoretical derivations, unfortunately, turned out to be incorrect (Johannesson, Pliskin, & Weinstein 1993, Wakker 1996). The intuition underlying their approach is valuable and the present paper may help formalize that intuition.

4 Theory

Like the other papers on the gambling effect, we restrict attention to decision under risk with given probabilities, hence we use the term lottery instead of the term gamble. A *lottery* $(p_1, x_1; \dots; p_n, x_n)$ yields *outcome* x_j with probability p_j , $j = 1, \dots, n$. Probabilities are nonnegative and sum to one. \mathcal{C} denotes the set of all conceivable outcomes and \mathcal{L} the set of all lotteries. We do not impose any condition on \mathcal{C} and it can be any arbitrary set, such as health states, commodity bundles, or monetary rewards. \mathcal{L} contains all finite probability distributions over \mathcal{C} . That is, every lottery is assumed to take only finitely many outcomes. Any lottery can be written as $(p_1, x_1; \dots; p_n, x_n)$ for a finite n . *Preferences* over lotteries are denoted by \succsim , with \succ (strict preference) and \sim (indifference) as usual.

Throughout, any outcome x is identified with the corresponding riskless lottery $(1, x)$. The set of all riskless (“Safe”) lotteries is identified with the outcome set \mathcal{C} . The set of the remaining, *risky*, lotteries, $\mathcal{L} - \mathcal{S}$, is denoted by \mathcal{R} . This set contains all lotteries $(p_1, x_1; \dots; p_n, x_n)$ with $n \geq 2$ and $x_i \neq x_j$ for some i, j with $p_i > 0$ and $p_j > 0$.

An *evaluation* or *representation* V is a function on the lotteries that determines preference, i.e., $P \succsim Q$ if and only if $V(P) \geq V(Q)$.

DEFINITION 1 The *gambling effect model* holds if there exists a *utility function* $u : \mathcal{C} \rightarrow \mathbb{R}$, a *cost function* $c : \mathcal{C} \rightarrow \mathbb{R}$, a *value function* $v = u - c$, and an evaluation V that assigns to each risky lottery $(p_1, x_1; \dots; p_n, x_n)$ its u expectation $p_1 u(x_1) + \dots + p_n u(x_n)$, and $v(x)$ to each outcome x . \square

At this moment we do not yet impose any restriction on the relations between v and u . These functions may order riskless outcomes differently. This issue is further discussed in the following section. We study preference conditions for the

above model. *Weak ordering* means that \succsim is *complete* ($P \succsim Q$ or $Q \succsim P$ for all lotteries P, Q) and transitive. For the independence condition we consider mixtures of lotteries. For lotteries P and Q and $0 \leq \lambda \leq 1$, $\lambda P + (1 - \lambda)Q$ is the lottery assigning probability $\lambda P(x) + (1 - \lambda)Q(x)$ to each outcome x . Here P and Q can be risky or riskless. The main weakening of expected utility is that independence is now imposed only on the risky lotteries, as in Fishburn (1980) and Schmidt (1998).

DEFINITION 2 *Gambling independence* holds if

$$P \succ Q \text{ implies } \lambda P + (1 - \lambda)R \succ \lambda Q + (1 - \lambda)R$$

for all risky P, Q, R and $0 < \lambda < 1$.

This axiom provides the most straightforward modification of expected utility to accommodate the Allais paradox. One simply adheres to expected utility except when the certainty effect can play a role. Burks (1977) suggested this approach normatively. He favored satisfying expected utility except in choice situations like Allais' paradox, where he preferred deviating from expected utility. His viewpoint is similar to the model characterized next. Observation 7 will demonstrate that Burks' approach faces its own normative problems.

To obtain real-valued evaluations, Archimedean axioms have to be imposed. These are usually somewhat complex and are "technical," i.e., have no direct empirical content. Hence the following axiom can be skipped by readers not interested in mathematical details. In our model the Archimedean axiom is more complicated than under expected utility because the riskless lotteries have to be treated separately. Our axiom generalizes Fishburn's (1980) Archimedean axioms. In Schmidt (1998), the Archimedean axiom is implied by topological assumptions. Condition (i) in Definition 3 restricts the traditional Archimedean axiom of expected utility to the case of risky

P, R so as to mix only risky lotteries. It covers the “regular” outcomes, i.e., outcomes that are not superior or inferior to all lotteries. Additional conditions have to be formulated for the “nonregular” outcomes that are preferred or dispreferred to all lotteries. These conditions are specified in (ii) and (iii).

DEFINITION 3 The *Archimedean Axiom* holds if:

- (i) For all lotteries Q and risky lotteries P, R , if $P \succ Q \succ R$ then $\lambda R + (1 - \lambda)P \succ Q$ and $Q \succ \mu P + (1 - \mu)R$ for some $0 < \lambda < 1$ and $0 < \mu < 1$.
- (ii) If, for some outcome x , $x \succ S$ ($x \prec S$) for all risky lotteries S , then μ (λ) in (i) can be chosen independently of P (Q).
- (iii) There exists a countable subset D of outcomes that is “order-dense” in the sense that, for every preference $x \succ y$, $x \succ d \succ y$ for some $d \in D$.

□

It can be verified that all conditions formulated above are necessary for the gambling effect model. They are also sufficient, as the following theorem shows.

THEOREM 4 Let \succsim be a preference relation on the set \mathcal{L} of all finite probability distributions over a set \mathcal{C} . Then preferences are evaluated by a gambling effect model if and only if \succsim satisfies (i) weak ordering, (ii) gambling independence, and (iii) the Archimedean axiom. □

Proofs are presented in the appendix. The uniqueness in the theorem is standard but is somewhat complex to formulate due to the different functions and their interactions. In short, u is unique up to scale and location and v shares the same scale and location except for outcomes that are preferred or dispreferred to all lotteries. For the latter, v is ordinal. Details are as follows.

OBSERVATION 5 *The uniqueness results for Theorem 4 are as follows. Assuming that u is replaced by another function u^* , we have:*

- (i) *There exist a real τ and a positive σ such that $u^* = \tau + \sigma \times u$ (u is unique up to scale and location).*
- (ii) *Given τ and σ as in (i), v is replaced by $v^* = \tau + \sigma \times v$ for all outcomes indifferent to some lottery.*
- (iii) *If outcomes exist that are preferred to all lotteries then v is ordinal there, i.e., it can be replaced by v^* if and only if v^* exceeds all expectations of u^* and further v^* is a strictly increasing transform of v .*
- (iv) *If outcomes exist that are dispreferred to all lotteries then v is ordinal there, i.e., it can be replaced by v^* if and only if v^* is exceeded by all expectations of u^* and further v^* is a strictly increasing transform of v .*
- (v) *c^* is defined accordingly, as $v^* - u^*$.*

□

It is remarkable that, outside the outcomes preferred or dispreferred to all lotteries, v is also cardinal (unique up to scale and location) so that diminishing marginal value can be defined and empirically verified. It can be distinguished from diminishing marginal utility in terms of u .

5 Stochastic Dominance

The analysis in the preceding section has not imposed any restriction on the relations between v and u . This section studies the case where u and v order outcomes in the same manner. This is called *ordinal equivalence*, and is formally defined as $u(x) \geq$

$u(y)$ if and only if $v(x) \geq v(y)$. It is well-known that ordinal equivalence holds if and only if $u(x) = f(v(x))$ for a strictly increasing function f . The condition is natural for monetary outcomes, with higher amounts preferred to lower amounts both under v and u , and was assumed by Fishburn (1980). For general outcomes, e.g., multiattribute outcomes or commodity bundles, the condition is not self-evident because the tradeoffs made between commodities may be different under risk than under certainty.

An example from health economics is as follows. Assume a two-attribute setting of chronic health states. One dimension designates duration, the other state of health. Assume that $x = (25, B)$, designating 25 years of life while being blind, followed by death. In the *time tradeoff* method for measuring value, introduced by Torrance, Thomas, & Sackett (1972), subjects are asked how many years of life they would be willing to sacrifice, hypothetically, to obtain full health. Say the subject says five years, meaning that $x \sim y$ with $y = (20, H)$, 20 years in perfect health followed by death. The subject may use an ordinally different evaluation system for risky decisions. Due to the gambling effect, a medical treatment that with some probability results in $(25, B)$ need not be indifferent to another treatment that results in the same probability distribution with, however, outcome $(25, B)$ replaced by outcome $(20, H)$. Such phenomena are well-known in health economics and have led Gafni and colleagues to develop the healthy years equivalent method.

In the absence of ordinal equivalence, there is no natural way to impose or even define stochastic dominance. We now turn to a preference condition that does ensure ordinal equivalence and next discuss stochastic dominance. *Gamble monotonicity* holds if replacement of an outcome by a preferred outcome always leads to a preferred lottery, assuming that both lotteries are risky. Formally, for all $0 < \lambda < 1$ and for all outcomes x, y and risky lotteries P , $x \succ y$ if and only if $\lambda x + (1 - \lambda)P \succ \lambda y + (1 - \lambda)P$.

This is similar to Fishburn's (1980) Axiom A2a. Note the restriction to risky lotteries P , which serves to avoid confounding with the gambling effect.

OBSERVATION 6 *Assume the gambling effect model. Ordinal equivalence of u and v holds if and only if gamble monotonicity holds.* \square

Under ordinal equivalence the riskless ordering of outcomes is relevant to risky choice, and stochastic dominance becomes meaningful. In our model, it is trivially satisfied for risky lotteries because there expected utility holds. For riskless lotteries, however, the condition becomes non-trivial due to the gambling effect. The formulation of the condition chosen here highlights the relations and differences with ordinal equivalence. Let us emphasize that the following formulation of stochastic dominance is logically equivalent, on our domain \mathcal{L} of finite probability distributions, to traditional formulations in terms of pointwise dominance of distribution functions.

Stochastic dominance holds if, for all outcomes x, y , $0 < \lambda < 1$, and lotteries P , $x \succcurlyeq y$ implies $\lambda x + (1 - \lambda)P \succcurlyeq \lambda y + (1 - \lambda)P$. The gambling effect model cannot satisfy this condition unless it reduces to expected utility. This was demonstrated for monetary outcomes and continuous nondecreasing u by Schmidt (1998, Proposition 1). We extend his result to the general gambling effect model.

OBSERVATION 7 *Under the gambling effect model, stochastic dominance holds if and only if expected utility is satisfied (i.e., $v = u$ can be chosen).* \square

Assuming that stochastic dominance is normatively desirable, the result shows that the gambling effect model is not normative. The interest of the model is descriptive and lies in its psychological plausibility.

6 Applications

In this section we assume the gambling effect model. We discuss the empirical measurement of its primitives and some restrictions.

Eliciting u and v from preferences

In the trivial case where all risky lotteries are indifferent, $u(x)$ is 0 for all outcomes x . We assume henceforth that not all lotteries are indifferent so that u is not constant. We take two arbitrary outcomes M and m such that $u(M) > u(m)$ and normalize $u(M) = 1$ and $u(m) = 0$. For instance, M may be a maximally and m a minimally conceivable monetary reward, or, in the health domain, M may be perfect health and m immediate death. Here are some ways for measuring u .

$$u(x) = \mu \text{ for some } 0 \leq \mu < 1 \text{ if } x \neq M \text{ and } (\tfrac{1}{2}, M; \tfrac{1}{2}, x) \sim (\tfrac{1}{2} + \tfrac{\mu}{2}, M; \tfrac{1}{2} - \tfrac{\mu}{2}, m).$$

$$u(x) = \mu \text{ for some } 0 < \mu \leq 1 \text{ if } x \neq m \text{ and } (\tfrac{1}{2}, x; \tfrac{1}{2}, m) \sim (\tfrac{\mu}{2}, M; 1 - \tfrac{\mu}{2}, m).$$

$$u(x) = \mu \text{ for some } 0 \leq \mu \leq 1 \text{ if } (\tfrac{1}{3}, M; \tfrac{1}{3}, m; \tfrac{1}{3}, x) \sim (\tfrac{1}{3} + \tfrac{\mu}{3}, M; \tfrac{2}{3} - \tfrac{\mu}{3}, m).$$

$$u(x) = \mu > 1 \text{ if } x \neq m \text{ and } (\tfrac{1}{2}, M; \tfrac{1}{2}, m) \sim (\tfrac{1}{2\mu}, x; 1 - \tfrac{1}{2\mu}, m).$$

$$u(x) = \mu < 0 \text{ if } x \neq M \text{ and } (\tfrac{1}{2}, M; \tfrac{1}{2}, m) \sim (1 - \tfrac{1}{2-2\mu}, M; \tfrac{1}{2-2\mu}, x).$$

Obviously, these are all methods for measuring traditional von Neumann-Morgenstern utility functions without invoking riskless options. McCord & de Neufville (1986) have argued for such measurements, precisely to avoid the certainty effect. Before, Davidson & Suppes (1956, p. 266) and Davidson, Suppes, & Siegel (1957, p. 18) also argued for such measurements with the explicit purpose to avoid or reduce distortions due to a specific taste or distaste for gambling. The gambling effect model provides a formal argument in favor of these proposals. Next we turn to the measurement of v .

If $x \sim P$ for a risky lottery P with expected utility μ , then $v(x) = \mu$. Hence we get:

If $u(M) > v(x) > u(m)$, then we can find $v(x) = \mu$ from a traditional “standard gamble” indifference $x \sim (\mu, M; 1 - \mu, m)$.

If there are outcomes x strictly preferred to each lottery, then u is bounded (see the proof of Theorem 4 in the appendix). On this set of strictly preferred outcomes x , the function v is only ordinally determined as long as it exceeds all u values. Similarly, for outcomes x strictly dispreferred to all risky lotteries, v is ordinally determined as long as it is below all u values.

A useful feature, distinguishing the gambling effect model from other models separating between risky and riskless utility, is that v can be directly revealed from risky choices, that is, from indifferences between risky and riskless options. It has been found empirically that measurements comparing risky to riskless options obtain more concave utility functions than measurements that only invoke risky options (McCord & de Neufville 1986, Wakker & Deneffe 1996). That may partly be due to the gambling effect, i.e., to an intrinsic cost of risk.

Constant gambling effect

The utility measurements can reveal special properties of u and v , and can be used to characterize them. One special case of interest concerns a constant cost function c . Then there is a fixed cost c for resorting to risk attitude and this cost is incurred whenever risk is perceived. The cost is independent of what the precise risk is. Riskless outcomes x are valued by $v(x)$, risky lotteries are evaluated by the expected utility of the function $u(x) = v(x) - c$. The utility measurement techniques, described in the preceding section, can reveal such a constant cost function.

Alternatively, the case can be identified through direct preference conditions. This was done by Fishburn (1980, Theorem 4). We adapt his condition to our context. First assume that $x \sim P$ and $y \sim Q$ for outcomes x, y and risky lotteries P, Q . Consider the mixtures $\frac{1}{2}x + \frac{1}{2}Q$ and $\frac{1}{2}y + \frac{1}{2}P$. Both mixtures are risky hence are evaluated by expected utility. It can be seen that the evaluation of each comprises half times: $v(x)$ plus $v(y)$ plus the cost c . Hence the mixtures must be indifferent. This leads to the following characterization, where we assume an indifference to avoid triviality and some pathological cases.

THEOREM 8 *Assume the gambling effect model and assume that $y \sim Q$ for an outcome y and a risky lottery Q . Then the cost function $c = v - u$ is constant if and only if the following conditions hold (hereafter, x and y are outcomes and P and Q are risky lotteries).*

(i) *Gamble monotonicity.*

(ii) *If $x \sim P$ and $y \sim Q$, then $\frac{1}{2}x + \frac{1}{2}Q \sim \frac{1}{2}y + \frac{1}{2}P$.*

(iii) *If $x \succ P$ for all risky lotteries P and $y \sim Q$, then, for all risky lotteries P , $\frac{1}{2}x + \frac{1}{2}Q \succ \frac{1}{2}y + \frac{1}{2}P$.*

(iv) *If $x \prec P$ for all risky lotteries P and $y \sim Q$, then, for all risky lotteries P , $\frac{1}{2}x + \frac{1}{2}Q \prec \frac{1}{2}y + \frac{1}{2}P$.*

□

Intransitivity instead of violation of stochastic dominance

Imagine a choice between a sure outcome x and a risky lottery P , such that each outcome of P is strictly preferred to x , but the gambling effect model assigns a higher value to x . As shown in Observation 7, such violations of stochastic dominance or

reversed violations with a preference for a dominated gamble, exist. Will a subject, directly having to choose from these two options, really forego a lottery that can only bring better outcomes? The answer depends on the context. The empirical findings of violation of stochastic dominance cited in the introduction, always concern framings of problems in which the stochastic dominance does not come out clearly and subjects do not realize it. Another reason for the gambling effect can lie in transaction costs; examples were given in Section 2. Alternatively, the subject may want to lay down his future plans and then forget about it, and simply does not even want to think about possible future profits if they are uncertain and small. Whenever there are such concrete reasons at the background of the cost function, it seems plausible that indeed the sure outcome x is chosen and stochastic dominance, defined in a narrow sense, is violated.

The case is different when the gambling effect is due to irrationalities and simplistic decision heuristics. When there is a clear dominance, it is plausible that subjects go by that dominance and their behavior is not described by the gambling effect model. Only if there is no clear dominance then subjects' behavior is described by the gambling effect model. Subjects do not realize that their evaluation entails indirect violations of monotonicity. They may prefer x to some lottery Q and Q to P whereas P dominates x . Such “editing” (Kahneman & Tversky 1979) entails violations of transitivity but constitutes an empirically plausible variation of the gambling effect model.

Certainty Preference

An interesting special case is the case where $c(x) \geq 0$ for all outcomes x , i.e., the cost of gambling is always nonnegative and there is a preference for certainty. Schmidt (1998) characterized this special case by the following preference condition:

For all risky lotteries P, Q , outcomes x , and $0 < \lambda < 1$, if $\lambda x + (1 - \lambda)Q \sim \lambda P + (1 - \lambda)Q$, then $x \succsim P$.

The proof that this condition is necessary and sufficient for nonnegative costs of gambling follows from substitution and is not elaborated here (see Schmidt 1998, Corollary 2).

A linear cost of gambling

We assumed in this paper the special case of Eq. 1 with C depending only on the sure outcome x . This can be rewritten, given the assumption throughout that W is expected utility, as a special case of dependency of C on the lottery P . To this effect, let W^* be expected utility with respect to v instead of u , and for each lottery P define $C^*(P)$ as the expectation of $v - u$. With these substitutions, $W^*(x) = v(x)$ for all x and $W^*(P) - C^*(P) = W(P)$ for all P , so that the representation is identical to the original one. Conversely, every case of Eq. 1 with C depending on P in a linear manner can, by inverse substitutions, be carried into a case of Eq. 1 with costs depending only on x .

7 Conclusion

Throughout the history of risky choice, researchers have been aware of the gambling effect, indicating that people use a different method for evaluating riskless options than for evaluating risky options. Theoretical models for the effect are almost absent, probably due to the holistic nature of the gambling effect and the entailed violation of stochastic dominance. Only recently, decision scientists have developed an interest in such basic violations of rationality conditions as stochastic dominance.

The model studied in this paper provides a tractable theoretical basis for the

gambling effect and describes, in a way, the most efficient deviation from expected utility to explain the Allais paradox. Tractable methods for the measurement of its primitives have been presented, in particular for measuring riskless utility at a cardinal level. The model sheds new light on risk aversion and its applications to gambling, insurance, and other contexts. It seems plausible that many investigations into risk attitudes have been confounded by the gambling effect.

Appendix A: Proofs

PROOF OF THEOREM 4 AND OBSERVATION 5. We first assume the gambling effect model and derive the preference conditions. Weak ordering is immediate, and gambling independence follows from the expected utility representation on \mathcal{R} . We finally demonstrate the Archimedean axiom. Part (i) follows from linearity of expected utility on \mathcal{R} and by taking λ and μ sufficiently small. Part (iii) follows because \succsim has a real-valued representation, v , on \mathcal{C} (Fishburn 1970). Finally, we turn to part (ii). If, for some outcome x , $x \succ S$ for all risky lotteries S , then $v(x)$ provides an upper bound for u , implying that μ in (i) can be chosen independently from P . The case of an x inferior to all risky lotteries is treated similarly, now with $v(x)$ a lower bound for u . All preference conditions have been satisfied.

We next assume the preference conditions and derive the gambling model effect. On \mathcal{R} all preference axioms of expected utility are satisfied and hence an expected utility representation can be obtained there, with the utility function denoted u . This is proved by Fishburn (1980, p. 438). We briefly sketch the proof. \mathcal{R} is a mixture set on which weak ordering, independence, and the appropriate Archimedean axiom (our Part (i) only for risky Q) imply a linear representation by Fishburn (1970, Theorem

8.4). Although no outcomes are contained in \mathcal{R} , it is still possible to define u on the outcomes such that the linear functional is expected utility with respect to u . E.g., $u(y) = \lambda u(x) + (1 - \lambda)u(z)$ can be elicited through indifference $\frac{1}{2}P + \frac{1}{2}y \sim \frac{1}{2}P + \frac{\lambda}{2}x + \frac{1-\lambda}{2}z$ for any risky lottery P . The results of Fishburn (1980 p. 438) as well as the indifference just written imply that u is unique up to scale and location.

Outcomes x such that $x \succ P$ for all $P \in \mathcal{R}$ are called *superior* outcomes, outcomes x such that $x \prec P$ for all $P \in \mathcal{R}$ are called *inferior* outcomes. We first extend the evaluation to outcomes that are neither superior nor inferior. For these outcomes, there exist risky lotteries P, Q such that $P \succ x \succ Q$. It can be derived from (i) of the Archimedean axiom that there exists a risky lottery R indifferent to x . This reasoning is similar to Fishburn (1970, C2 in Theorem 8.3). In short, if neither P nor Q can play the role of R , then it follows from gambling independence that $\{\lambda \in [0, 1] : \lambda P + (1 - \lambda)Q \succ x\}$ is convex, and from (i) of the Archimedean axiom that it is of the form $(\mu, 1]$. Similarly, $\{\lambda \in [0, 1] : \lambda P + (1 - \lambda)Q \prec x\}$ is of the form $[0, \nu)$. It then follows that $R = \mu P + (1 - \mu)Q$ can be taken (it can also be seen that $\mu = \nu$). We define $v(x)$ as the expected utility of R . This is the only definition of v possible, given u . We have now established the gambling effect model, and its uniqueness, on the union of all risky lotteries and all outcomes that are neither inferior nor superior.

Next we consider superior and inferior outcomes. Let there exist a superior outcome x . Then u must be bounded above, due to (ii) of the Archimedean axiom. To wit, assume that u is not constant. Then we can take two risky lotteries Q, R with $Q \succ R$. Assume there is a risky P with $P \succ Q$, if such P does not exist then u is bounded and we are done. Take μ as in (ii) of the Archimedean axiom. Writing EU for expected utility, it follows that $\mu EU(P) < EU(Q) - (1 - \mu)EU(R)$ which provides an upper bound to $EU(P)$ and thus u must be bounded from above. Similarly,

if there exist inferior outcomes, then u must be bounded from below. By part (iii) of the Archimedean axiom and Fishburn (1970), there exists a function v^* on \mathcal{C} that represents \succsim on \mathcal{C} . On the superior outcomes, we must and can let v be any ordinal transform of v^* that exceeds the upper bound of u , on the inferior outcomes we must and can let v be any ordinal transform of v^* that is below the lower bound of u . This establishes the gambling effect model and also the uniqueness results of Observation 5. \square

PROOF OF OBSERVATION 6.

First assume ordinal equivalence of u and v . Assume that we replace a positive-probability outcome in a lottery by a strictly preferred outcome in such a manner that the lottery is risky both before and after the substitution. It means that we have replaced the outcome by one with a strictly higher v value, hence, by ordinal equivalence, by one with a strictly higher u value. Given positiveness of the probability, the replacement strictly increases the expected utility of the lottery, hence its preference value. A similar reasoning applies if we replace an outcome by an indifferent outcome, or by a strictly dispreferred outcome. From this and weak ordering, gamble monotonicity follows.

Next assume that gamble monotonicity holds. Consider two outcomes x, y with $v(x) > v(y)$. This implies $x \succ y$. Let P be any risky lottery. By gamble monotonicity, $0.5x + 0.5P \succ 0.5y + 0.5P$. Substitution of expected utility implies $u(x) > u(y)$. Similarly, $v(x) = v(y)$ implies $u(x) = u(y)$ and $v(x) < v(y)$ implies $u(x) < u(y)$. That is, v and u are ordinally equivalent. Notice that in this step we only used gamble monotonicity with $\lambda = 0.5$, hence it would have sufficed to require the condition only for that λ . This was actually the formulation used by Fishburn (1980, Axiom A2a).

\square

PROOF OF OBSERVATION 7.

It is obvious that expected utility implies stochastic dominance. Hence we assume stochastic dominance and the gambling effect model, and derive expected utility.

CLAIM 1. If for outcome x , there exists an outcome $y \succ x$, then $v(x) \leq u(x)$.

Proof. Consider a lottery $(p, y; 1 - p, x)$. By stochastic dominance, the lottery is preferred to x , hence its expected utility exceeds $v(x)$. For p tending to zero, the expected utility tends to $u(x)$, hence $u(x)$ exceeds $v(x)$. *QED*

Similarly it can be demonstrated that:

CLAIM 2. If for outcome x , there exists an outcome $y \prec x$, then $v(x) \geq u(x)$.

The two claims show that $u(x) = v(x)$ for all outcomes x , except best or worst outcomes. Consider now a best outcome x . To avoid triviality we assume two non-indifferent outcomes, hence, by Claim 2, $v(x) \geq u(x)$. If $v(x) > u(x)$, then we can simply redefine $v(x) = u(x)$ and $v(y) = u(y)$ for all outcomes $y \sim x$. This definition leads to correct descriptions of all preferences with x involved. We have strict preference and strictly higher value for x than for each lottery assigning positive probability to any outcome strictly worse than x . We have indifference and equal value for x and each lottery assigning probability 1 to outcomes indifferent to x . Similarly, for a worst outcome y we can redefine $v(y) = u(y)$. \square

PROOF OF THEOREM 8.

Some explanation was already given above the theorem. Necessity of the preference conditions follows from substitution. For sufficiency, assume the preference

conditions. Take y as assumed in the theorem. Define $\gamma = c(y)$. On the set of nonsuperior and noninferior outcomes x , $c(x) = \gamma$ follows from condition (ii) and substitution. Define $v^* = u + \gamma$, also for superior and inferior outcomes.

If x is superior then condition (iii) ensures that $v(x) > EU(P)$ for all lotteries. If x is inferior then condition (iv) ensures that $v(x) < EU(P)$ for all lotteries. Now the representation accommodates all preferences. \square

References

- Birnbaum, Michael H. & Juan B. Navarrete (1998), “Testing Descriptive Utility Theories: Violations of Stochastic Dominance and Cumulative Independence,” *Journal of Risk and Uncertainty* 17, 49–78.
- Bombardier, C., A.D. Wolfson, A.J. Sinclair, & A. McGreer (1982), “Comparison of Three Preference Measurement Methodologies in the Evaluation of a Functional Status Index.” In Raisa B. Deber & G. Thompson (Eds.), *Choices in Health Care: Decision Making and Evaluation of Effectiveness*, University of Toronto.
- Burks, Arthur W. (1977), “*Chance, Cause, Reason (An Inquiry into the Nature of Scientific Evidence)*.” The University of Chicago Press, Chicago.
- Conlisk, John (1989), “Three Variants on the Allais Example,” *American Economic Review* 79, 392–407.
- Conlisk, John (1993), “The Utility of Gambling,” *Journal of Risk and Uncertainty* 6, 255–275.
- Davidson, Donald & Patrick Suppes (1956), “A Finitistic Axiomatization of Utility and Subjective Probability,” *Econometrica* 24, 264–275.
- Davidson, Donald, Patrick Suppes, & Sidney Siegel (1957), “*Decision Making: An Ex-*

- perimental Approach.*” Stanford University Press, Stanford; Chapter 2 reprinted in Ward Edwards & Amos Tversky (1967, Eds.), *Decision Making*, 170–207, Penguin, Harmondsworth.
- Dekel, Eddie (1986), “An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom,” *Journal of Economic Theory* 40, 304–318.
- Dyer, James S. & Rakesh K. Sarin (1982), “Relative Risk Aversion,” *Management Science* 28, 875–886.
- Edwards, Ward (1962), “Subjective Probabilities Inferred from Decisions,” *Psychological Review* 69, 109–135.
- Fishburn, Peter C. (1970), *Utility Theory for Decision Making.* Wiley, New York.
- Fishburn, Peter C. (1980), “A Simple Model for the Utility of Gambling,” *Psychometrika* 45, 435–448.
- Fishburn, Peter C. (1992), “On Nonstandard Nontransitive Additive Utility,” *Journal of Economic Theory* 56, 426–433.
- Gafni, Amiram & Stephen Birch (1997), “QALYs and HYE; Spotting the Differences,” *Journal of Health Economics* 16, 601–608.
- Gafni, Amiram & George W. Torrance (1984), “Risk Attitude and Time Preference in Health,” *Management Science* 30, 440–451.
- Grant, Simon, Atsushi Kajii, & Ben Polak (1998), “Intrinsic Preference for Information,” *Journal of Economic Theory* 83, 233–259.
- Harless, David W. & Colin F. Camerer (1994), “The Predictive Utility of Generalized Expected Utility Theories,” *Econometrica* 62, 1251–1289.
- Humphrey, Stephen J. (1998), “More Mixed Results on Boundary Effects,” *Economics Letters* 61, 79–84.
- Johannesson, Magnus, Joseph S. Pliskin, & Milton C. Weinstein (1993), “Are Healthy-

- Years Equivalents an Improvement over Quality-Adjusted Life Years?”, *Medical Decision Making* 13, 281–286.
- Kahneman, Daniel & Amos Tversky (1979), “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica* 47, 263–291.
- Le Menestrel, Marc (1999), “A Process Approach to the Utility of Gambling,” INSEAD, Fontainebleau, France.
- Lichtenstein, Sarah & Paul Slovic (1971), “Reversals of Preference between Bids and Choices in Gambling Decisions,” *Journal of Experimental Psychology* 89, 46–55.
- Lindman, Harold R. (1971), “Inconsistent Preferences among Gambles,” *Journal of Experimental Psychology* 89, 390–397.
- Loomes, Graham (1993), “Disparities between Health State Measures: An Explanation and Some Implications.” In W. Gerrard (Ed.), *The Economics of Rationality*, 149–178 (Chapter 9), Routledge, London.
- Luce, R. Duncan (1991), “Rank- and-Sign Dependent Linear Utility Models for Binary Gambles,” *Journal of Economic Theory* 53, 75–100.
- Luce, R. Duncan (1999), “*Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*.” Forthcoming.
- Luce, R. Duncan & Anthony A.J. Marley (1999), “On Elements of Chance,” in preparation.
- McCord, Mark R. & Richard de Neufville (1986), ““Lottery Equivalents”: Reduction of the Certainty Effect Problem in Utility Assessment,” *Management Science* 32, 56–60.
- Mehrez, Abraham & Amiram Gafni (1989), “Quality-Adjusted Life-Years, Utility Theory and Healthy Years Equivalents,” *Medical Decision Making* 9, 142–149.
- Neilson, W.S. (1992), “Some Mixed Results on Boundary Effects,” *Economics Letters* 39, 275–278.

- Pope, Robin E. (1995), "Towards a More Precise Decision Framework; A Separation of the Negative Utility of Chance from Diminishing Marginal Utility and the Preference for Safety," *Theory and Decision* 39, 241–265.
- Quiggin, John (1981), "Risk Perception and Risk Aversion among Australian Farmers," *Australian Journal of Agricultural Economics* 25, 160–169.
- Richardson, J. (1990), "Cost Utility Analysis: What Should Be Measured - Utility, Value or Healthy Year Equivalents?," National Centre for Health Program Evaluation, Monash University; Paper presented to the Second World Congress of Health Economics, University of Zurich, Switzerland, September 10-14, 1990.
- Royden, Halsey L., Patrick Suppes, & Karol Walsh (1959), "A Model for the Experimental Measurement of the Utility of Gambling," *Behavioral Science* 4, 11–18.
- Savage, Leonard J. (1954), "*The Foundations of Statistics*." Wiley, New York. (Second edition 1972, Dover, New York.)
- Schmidt, Ulrich (1998), "A Measurement of the Certainty Effect," *Journal of Mathematical Psychology* 42, 32–47.
- Skiadas, Costis (1997), "Subjective Probability under Additive Aggregation of Conditional Preferences," *Journal of Economic Theory* 76, 242–271.
- Sopher, Barry & Gary Gigliotti (1993), "A Test of Generalized Expected Utility Theory," *Theory and Decision* 35, 75–106.
- Stalmeier, Peep F.M. & Thom G.G. Bezembinder (1998), "The Discrepancy between Risky and Riskless Utilities: A Matter of Framing?," NICI, University of Nijmegen, The Netherlands.
- Starmer, Chris & Robert Sugden (1993), "Testing for Juxtaposition and Event-Splitting Effects," *Journal of Risk and Uncertainty* 6, 235–254.
- Stiggelbout, Anne M., Gwendoline M. Kiebert, Job Kievit, Jan-Willem H. Leer, G. Stoter, Hanneke C.J.M. de Haes (1994), "Utility Assessment In Cancer Patients:

- Adjustment of Time Tradeoff Scores for the Utility of Life Years and Comparison with Standard Gamble Scores,” *Medical Decision Making* 14, 82–90.
- Sugden, Robert (1993), “An Axiomatic Foundation for Regret Theory,” *Journal of Economic Theory* 60, 159–180.
- Torrance, George W., Warren H. Thomas, & David L. Sackett (1972), “A Utility Maximization Model for Evaluation of Health Care Programs,” *Health Services Research* 7, 118–133.
- Tversky, Amos (1967), “Additivity, Utility, and Subjective Probability,” *Journal of Mathematical Psychology* 4, 175–201.
- Tversky, Amos & Daniel Kahneman (1981), “The Framing of Decisions and the Psychology of Choice,” *Science* 211, 453–458.
- Tversky, Amos & Daniel Kahneman (1986), “Rational Choice and the Framing of Decisions,” *Journal of Business* 59, S251–S278.
- Tversky, Amos & Daniel Kahneman (1992), “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty* 5, 297–323.
- Viscusi, W. Kip (1989), “Prospective Reference Theory: Toward an Explanation of the Paradoxes,” *Journal of Risk and Uncertainty* 2, 235–264.
- von Neumann, John & Oskar Morgenstern (1944, 1947, 1953), “*Theory of Games and Economic Behavior*.” Princeton University Press, Princeton NJ.
- Wakker, Peter P. (1994), “Separating Marginal Utility and Probabilistic Risk Aversion,” *Theory and Decision* 36, 1–44.
- Wakker, Peter P. (1996), “A Criticism of Healthy-Years Equivalents,” *Medical Decision Making* 16, 207–214.
- Wakker, Peter P. & Daniel Deneffe (1996), “Eliciting von Neumann-Morgenstern Utilities when Probabilities Are Distorted or Unknown,” *Management Science* 42, 1131–1150.